

The time-fractional radiative transport equation – Continuous-time random walk, diffusion approximation, and Legendre-polynomial expansion

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We consider the radiative transport equation in which the time derivative is replaced by the Caputo derivative. Such fractional-order derivatives are related to anomalous transport and anomalous diffusion. In this paper we describe how the time-fractional radiative transport equation is obtained from continuous-time random walk and see how the equation is related to the time-fractional diffusion equation in the asymptotic limit. Then we solve the equation with Legendre-polynomial expansion.

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I. INTRODUCTION

Anomalous diffusion is often observed in nature^{31,33}. For example, tracer particles flowing in an aquifer exhibits anomalous diffusion¹. At the macroscopic scale after multiple scattering takes place, such anomalous diffusion is governed by fractional diffusion equations^{31,32,39}. Considering the fact that the diffusion equation appears in the asymptotic limit of the radiative transport equation or the linear Boltzmann equation¹⁸, one can expect that at the mesoscopic scale there exist anomalous transport phenomena which are described by the fractional radiative transport equation. The use of the radiative transport equation was proposed for predicting the concentration of radionuclides in fractured rock underground^{40,41}. If this happens, then its fractional version must appear just like the fractional diffusion equation shows up when the diffusion process takes place in a complex structure.

Let $\alpha \in (0, 1)$ and $\sigma_t, \sigma_s \in (0, \infty)$ be constants determined by the medium under consideration. We suppose $\sigma_t > \sigma_s$. Let $v > 0$ be a constant speed. Let $u(x, \mu, t)$ ($x \in \mathbb{R}$, $\mu \in [-1, 1]$, $t \in [0, \infty)$) be the angular density. We consider the following initial-value problem for the time-fractional radiative transport equation.

$$\begin{cases} \partial_t^\alpha u(x, \mu, t) + v\mu \partial_x u(x, \mu, t) + \sigma_t u(x, \mu, t) = \sigma_s \int_{-1}^1 p(\mu, \mu') u(x, \mu', t) d\mu', \\ u(x, \mu, 0) = \delta(x)\delta(\mu - \mu_0), \end{cases} \quad (1)$$

where $\delta(\cdot)$ is the Dirac delta function and ∂_t^α is the Caputo fractional derivative³, which is defined by³⁵

$$\partial_t^\alpha u(\cdot, \cdot, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{\partial_{t'} u(\cdot, \cdot, t')}{(t - t')^\alpha} dt', \quad 0 < \alpha < 1,$$

with $\Gamma(\cdot)$ the Gamma function. Indeed, u in (1) is the fundamental solution of the time-fractional radiative transport equation. We note that recently ∂_t^α was redefined more generally using fractional Sobolev spaces¹⁰. Compared with the Riemann-Liouville derivative, the Caputo derivative is not singular at $t = 0$. Thus we can have the same initial condition in (1) and in the corresponding equation of the first derivative ∂_t . The phase function $p(\mu, \mu')$ satisfies

$$\int_{-1}^1 p(\mu, \mu') d\mu' = 1, \quad \forall \mu \in [-1, 1].$$

Anomalous transport phenomena are in the transport regime when the distance of interest is not too large compared to the scattering mean free path v/σ_s , and as is shown

below, the time-fractional diffusion equation is obtained from (1) in the asymptotic limit. The time-fractional diffusion equation has been intensively studied. In addition to several examples^{31,33}, we point out that the behavior of water transport in granite was successfully reproduced by the random walk process with a power-law distribution¹¹. It is proposed that if there are two porosities, the mass transport in fractured porous aquifer should be governed by the diffusion equation in which both ∂_t and ∂_t^α appear⁶. The Cauchy problem⁴ and initial-boundary-value problem^{24,26} were considered for the time-fractional diffusion equation. The maximum principle was established²³. The technique of eigenfunction expansion was developed³⁷. Numerical algorithms for the equation have been developed²². Moreover the standard time-fractional diffusion equation was generalized to equations with multiple Caputo derivatives^{19,25} and distributed-order equations^{16,20}. See the recent review by Jin and Rundell¹⁴.

The rest of the paper is organized as follows. In §II, we obtain the time-fractional radiative transport equation from continuous-time random walk. In §III, we see that the time-fractional diffusion equation emerges from the time-fractional radiative transport equation when absorption is small, propagation distance is large, and observation time is long. In §IV, we express the solution to the time-fractional radiative transport equation in the form of Legendre polynomial expansion. In §V, we numerically compute the solutions of the time-fractional radiative transport equation and of the time-fractional diffusion equation. Finally in §VI, concluding remarks are made. The subtraction of the ballistic term is considered in Appendix.

II. CONTINUOUS-TIME RANDOM WALK

We consider the continuous-time random walk whose jump probability density function $\varphi(x, t; \mu, \mu')$ ($x \in \mathbb{R}$, $t \in [0, \infty)$, $\mu, \mu' \in [-1, 1]$) is given by

$$\varphi(x, t; \mu, \mu') = [\xi_s \delta(x) p(\mu, \mu') + (1 - \xi_t) \delta(x - \mu r) \delta(\mu - \mu')] w(t), \quad (2)$$

where $\xi_t \in (0, 1)$, $\xi_s \in (0, \xi_t)$, and $r > 0$ are some constants. The first term represents scattering and the second term in the square brackets of (2) is responsible for transport. The waiting time probability density function $w(t)$ is obtained as

$$(1 - \xi_a) w(t) = \int_{-1}^1 \int_{-\infty}^{\infty} \varphi(x, t; \mu, \mu') dx d\mu',$$

where $\xi_a = \xi_t - \xi_s > 0$ is the probability for absorption. The left-hand side of the above-mentioned equation shows the probability that the test particle is not absorbed in the medium and makes a jump after the time t .

Let $\eta(x, \mu, t)$ be the probability density function of just having arrived at position x at time t in direction μ . Let $P(x, \mu, t)$ be the probability density function of being at $(x, \mu, t) \in \mathbb{R} \times [-1, 1] \times [0, \infty)$. We consider the following continuous-time random walk process.

$$\begin{cases} \eta(x, \mu, t) = \int_0^t \int_{-1}^1 \int_{-\infty}^{\infty} \eta(x', \mu', t') \varphi(x - x', t - t'; \mu, \mu') dx' d\mu' dt' + a(x, \mu) \delta(t), \\ P(x, \mu, t) = \int_0^t \eta(x, \mu, t') \Phi(t - t') dt', \end{cases}$$

where $a(x, \mu)$ is the initial value which is a function of x and μ , $\Phi(t)$ is the cumulative probability of not having moved during t , which is given by

$$\Phi(t) = 1 - \int_0^t w(t') dt'.$$

By the Fourier-Laplace transform we have

$$\begin{aligned} (\mathcal{LFP})(k, \mu, s) &= \int_0^{\infty} e^{-st} \int_{-\infty}^{\infty} e^{-ikx} P(x, \mu, t) dx dt \\ &= (\mathcal{LF}\eta)(k, \mu, s)(\mathcal{L}\Phi)(s), \end{aligned}$$

where

$$(\mathcal{L}\Phi)(s) = \frac{1 - (\mathcal{L}w)(s)}{s}.$$

Hence we obtain

$$\begin{aligned} (\mathcal{LF}\eta)(k, \mu, s) &= \left[\xi_s \int_{-1}^1 p(\mu, \mu') (\mathcal{LF}\eta)(k, \mu', s) d\mu' \right. \\ &\quad \left. + (1 - \xi_t) (\mathcal{LF}\eta)(k, \mu, s) e^{-i\mu rk} \right] (\mathcal{L}w)(s) + (\mathcal{F}a)(k, \mu). \end{aligned}$$

We consider small k and use

$$e^{-i\mu rk} \sim 1 - i\mu rk.$$

Thus we arrive at

$$\begin{aligned} &\frac{1 - (\mathcal{L}w)(s)}{(\mathcal{L}w)(s)} \left[(\mathcal{LP})(x, \mu, s) - \frac{1}{s} P(x, \mu, 0) \right] \\ &= \xi_s \int_{-1}^1 p(\mu, \mu') (\mathcal{LP})(x, \mu', s) d\mu' - [\xi_t + (1 - \xi_t)r\mu\partial_x] (\mathcal{LP})(x, \mu, s). \end{aligned}$$

Recalling $0 < \alpha < 1$, we have^{35,38}

$$(\mathcal{L}\partial_t^\alpha f)(s) = s^\alpha (\mathcal{L}f)(s) - s^{\alpha-1}f(0).$$

Let us assume that the waiting time probability density function behaves as

$$(\mathcal{L}w)(s) \sim 1 - (\tau s)^\alpha, \quad 0 < s \ll \frac{1}{\tau}.$$

We introduce

$$\sigma_t = \frac{\xi_t}{\tau^\alpha}, \quad \sigma_s = \frac{\xi_s}{\tau^\alpha}, \quad v = \frac{(1 - \xi_t)r}{\tau^\alpha}.$$

We asymptotically obtain

$$\partial_t^\alpha P(x, \mu, t) + v\mu\partial_x P(x, \mu, t) + \sigma_t P(x, \mu, t) = \sigma_s \int_{-1}^1 p(\mu, \mu') P(x, \mu', t) d\mu'.$$

This is (1).

Remark II.1. In this section we implemented the effect of absorption in our random walk by introducing ξ_a . Such extension of the usual continuous-time random walk is done by Hornung, Berkowitz, and Barkai¹³, and by Henry, Langlands, and Wearne¹². Indeed, we arrive at the same conclusion by instead writing (2) as

$$\varphi(x, t; \mu, \mu') = [\xi_s \delta(x) p(\mu, \mu') + (1 - \xi_t) \delta(x - \mu r) \delta(\mu - \mu')] \frac{w(t)}{1 - \xi_a},$$

with the waiting time probability density function $w(t)$ introduced as

$$w(t) = \int_{-1}^1 \int_{-\infty}^{\infty} \varphi(x, t; \mu, \mu') dx d\mu'.$$

We can then give $\eta(x, \mu, t)$ and $P(x, \mu, t)$ as

$$\begin{cases} \eta(x, \mu, t) = (1 - \xi_a) \int_0^t \int_{-1}^1 \int_{-\infty}^{\infty} \eta(x', \mu', t') \varphi(x - x', t - t'; \mu, \mu') dx' d\mu' dt' + a(x, \mu) \delta(t), \\ P(x, \mu, t) = (1 - \xi_a) \int_0^t \eta(x, \mu, t') \Phi(t - t') dt'. \end{cases}$$

Note that $P(x, \mu, 0) = (1 - \xi_a)a(x, \mu)$. Thus the relation to the past work^{12,13} becomes clearer.

III. DIFFUSION APPROXIMATION

Let us suppose that the ratio $\epsilon > 0$ of the mean free path to the propagation distance is small. We scale t, x as $t \rightarrow \epsilon^{2/\alpha} t$ and $x \rightarrow \epsilon x$. Furthermore we scale $\sigma_a \rightarrow \sigma_a/\epsilon^2$ assuming σ_a is small (recall $\sigma_a = \sigma_t - \sigma_s$). Although the radiative transport equation (1) has the Caputo derivative, we obtain the time-fractional diffusion equation by following the standard procedure^{2,18,36}. In this section we assume that $p(\mu, \mu') = p(\mu', \mu)$. We can write the time-fractional radiative transport equation as

$$\epsilon^2 \partial_t^\alpha u(x, \mu, t) + \epsilon v \mu \partial_x u(x, \mu, t) + (\epsilon^2 \sigma_a + \sigma_s) u(x, \mu, t) = \sigma_s \int_{-1}^1 p(\mu, \mu') u(x, \mu', t) d\mu'.$$

We write

$$u(x, \mu, t) = U_{\text{DA}}(x, \mu, t) + \epsilon U_{\text{DA}}^{(1)}(x, \mu, t) + \epsilon^2 U_{\text{DA}}^{(2)}(x, \mu, t) + \dots.$$

Let us collect terms of order ϵ^0 . We obtain

$$\sigma_s U_{\text{DA}}(x, \mu, t) = \sigma_s \int_{-1}^1 p(\mu, \mu') U_{\text{DA}}(x, \mu', t) d\mu'.$$

The above equation implies that U_{DA} is independent of μ ; hereafter we write $U_{\text{DA}}(x, \mu, t) = U_{\text{DA}}(x, t)$. The terms of order ϵ^1 yields

$$v \mu \partial_x U_{\text{DA}}(x, t) + \sigma_s U_{\text{DA}}^{(1)}(x, \mu, t) = \sigma_s \int_{-1}^1 p(\mu, \mu') U_{\text{DA}}^{(1)}(x, \mu', t) d\mu'.$$

We obtain

$$U_{\text{DA}}^{(1)}(x, \mu, t) = -\frac{v}{(1-g)\sigma_s} \mu \partial_x U_{\text{DA}}(x, t),$$

where $g \in (-1, 1)$ satisfies

$$\mu g = \int_{-1}^1 \mu' p(\mu, \mu') d\mu'.$$

By collecting terms of order ϵ^2 we have

$$\begin{aligned} & \partial_t^\alpha U_{\text{DA}}(x, t) + \mu \partial_x U_{\text{DA}}^{(1)}(x, \mu, t) + \sigma_a U_{\text{DA}}(x, t) + \sigma_s U_{\text{DA}}^{(2)}(x, \mu, t) \\ &= \sigma_s \int_{-1}^1 p(\mu, \mu') U_{\text{DA}}^{(2)}(x, \mu', t) d\mu'. \end{aligned}$$

If we integrate the above equation over μ , we obtain

$$\partial_t^\alpha U_{\text{DA}}(x, t) - D_0 \partial_x^2 U_{\text{DA}}(x, t) + \sigma_a U_{\text{DA}}(x, t) = 0, \quad (3)$$

where

$$D_0 = \frac{v}{3(1-g)\sigma_s}. \quad (4)$$

Thus the time-fractional diffusion equation is obtained in the asymptotic limit of (1).

One remark needs to be made. We have the second derivative for the spatial variable x in (3). In a similar setting, it is known that the space-fractional diffusion equation is obtained if the phase function decays with power-law as a function of the speed of propagating particles^{29,30}.

IV. LEGENDRE-POLYNOMIAL EXPANSION

Let us suppose $p(\mu, \mu')$ is given by

$$p(\mu, \mu') = \frac{1}{2} \sum_{l=0}^L \beta_l P_l(\mu) P_l(\mu'),$$

where $L \geq 0$, and β_l ($l = 0, 1, \dots, L$) are positive constants such as $\beta_0 = 1$, $\beta_l < 2l + 1$ for $l \geq 1$. Here, $P_l(\mu)$ are the Legendre polynomials recursively given by

$$(l+1)P_{l+1}(\mu) = (2l+1)\mu P_l(\mu) - lP_{l-1}(\mu), \quad P_1(\mu) = \mu, \quad P_0(\mu) = 1, \quad \mu \in [-1, 1].$$

In the time-independent case, an analytical solution of the space-fractional radiative transport equation was found¹⁵. In this section we solve (1). Let us expand u with Legendre polynomials.

$$(\mathcal{F}u)(k, \mu, t) = \sum_{l=0}^{\infty} \sqrt{2l+1} c_l(k, t; \mu_0) P_l(\mu). \quad (5)$$

We perform the Fourier transform in (1) and substitute (5). We have

$$\begin{aligned} & (\partial_t^\alpha + ivk\mu + \sigma_t) \sum_{l=0}^{\infty} \sqrt{2l+1} c_l(k, t; \mu_0) P_l(\mu) \\ &= \sigma_s \sum_{l=0}^{\infty} \sqrt{2l+1} c_l(k, t; \mu_0) \frac{\beta_l}{2l+1} P_l(\mu) \Theta(L-l). \end{aligned}$$

Let us introduce

$$h_l = 2l + 1 - \frac{\sigma_s}{\sigma_t} \beta_l \Theta(L-l).$$

Let N ($\geq L$) be an integer. We take projections with $P_l(\mu)$ ($l = 0, 1, \dots, N$) and obtain

$$\frac{ivkl}{\sqrt{4l^2-1}} c_{l-1} + \frac{ivk(l+1)}{\sqrt{4(l+1)^2-1}} c_{l+1} + \partial_t^\alpha c_l + \frac{\sigma_t h_l}{2l+1} c_l = 0,$$

where we used the recurrence relations and orthogonality relations of Legendre polynomials,

$$\mu P_l(\mu) = \frac{l+1}{2l+1} P_{l+1}(\mu) + \frac{l}{2l+1} P_{l-1}(\mu), \quad (6)$$

and

$$\int_{-1}^1 P_l(\mu) P_{l'}(\mu) d\mu = \frac{2}{2l+1} \delta_{ll'}.$$

The above equation is expressed as

$$A(k) \mathbf{c}(k, t; \mu_0) + \partial_t^\alpha \mathbf{c}(k, t; \mu_0) = 0,$$

where $A(k)$ is an $(N+1) \times (N+1)$ matrix and $\mathbf{c}(k, t; \mu_0)$ is an $N+1$ dimensional vector defined by

$$\{A(k)\}_{ll'} = \frac{ivkl}{\sqrt{4l^2-1}} \delta_{l-1,l'} + \frac{\sigma_t h_l}{2l+1} \delta_{l,l'} + \frac{ivk(l+1)}{\sqrt{4(l+1)^2-1}} \delta_{l+1,l'}, \quad (7)$$

$$\{\mathbf{c}(k, t; \mu_0)\}_l = c_l(k, t; \mu_0). \quad (8)$$

When the Legendre polynomial expansion is used, tridiagonal matrices such as $A(k)$ appear due to the three-term recurrence relation (6)^{7,8,21,34}. By taking the Laplace transform we have

$$(\mathcal{L}\mathbf{c})(k, s; \mu_0) = (A(k) + s^\alpha)^{-1} s^{\alpha-1} \mathbf{c}(k, 0; \mu_0),$$

where we used

$$(\mathcal{L}\partial_t^\alpha \mathbf{c})(k, s; \mu_0) = s^\alpha (\mathcal{L}\mathbf{c})(k, s; \mu_0) - s^{\alpha-1} \mathbf{c}(k, 0; \mu_0), \quad 0 < \alpha \leq 1.$$

Let us recall that the Mittag-Leffler function is given by³⁵

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z, \alpha \in \mathbb{C}, \quad \Re \alpha > 0,$$

and the Laplace transform is obtained as

$$\mathcal{L}\{E_\alpha(z t^\alpha); s\} = \frac{s^{\alpha-1}}{s^\alpha - z}, \quad z, s, \alpha \in \mathbb{C}, \quad \Re s, \Re \alpha > 0, \quad \left| \frac{z}{s^\alpha} \right| < 1.$$

Thus we find

$$\mathbf{c}(k, t; \mu_0) = E_\alpha(-A(k)t^\alpha) \mathbf{c}(k, 0; \mu_0).$$

Since $\delta(\mu - \mu_0) = \sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(\mu) P_l(\mu_0)$, we obtain

$$\{\mathbf{c}(k, 0; \mu_0)\}_l = \frac{\sqrt{2l+1}}{2} P_l(\mu_0).$$

Let $\lambda_n(k)$ and $\mathbf{v}_n(k)$ be the n th eigenvalue and eigenvector of the matrix $A(k)$. We can write $A(k)$ as

$$A(k) = Q(k) D(k) Q(k)^{-1},$$

where

$$Q(k) = (\mathbf{v}_0(k) \ \mathbf{v}_1(k) \ \cdots \ \mathbf{v}_N(k)), \quad D(k) = \text{diag}(\lambda_0(k), \lambda_1(k), \dots, \lambda_N(k)).$$

We have

$$\{A(k)\}_{ij} = \{Q(k)D(k)Q(k)^{-1}\}_{ij} = \sum_{n=0}^N \lambda_n(k) v_n^{(i)}(k) v_n^{(j)*}(k),$$

where $v_n^{(i)}(k)$ is the i th component of $\mathbf{v}_n(k)$. Therefore we can write

$$\{\mathbf{c}(k, t; \mu_0)\}_l = \sum_{j=0}^N \frac{\sqrt{2j+1}}{2} P_j(\mu_0) \sum_{n=0}^N v_n^{(l)}(k) v_n^{(j)*}(k) E_\alpha(-\lambda_n(k)t^\alpha).$$

Noting (8), Eq. (5) yields

$$\begin{aligned} u(x, \mu, t) &\approx u(x, \mu, t; N) \\ &:= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \sum_{l=0}^N \sqrt{2l+1} c_l(k, t; \mu_0) P_l(\mu) dk. \end{aligned} \quad (9)$$

Since k appears always as ik , we see

$$c_l(-k, t; \mu_0) = c_l(k, t; \mu_0)^*.$$

We obtain

$$\begin{aligned} u(x, \mu, t; N) &= \sum_{l=0}^N \frac{\sqrt{2l+1}}{\pi} P_l(\mu) \\ &\times \int_0^\infty [\cos(kx) \Re c_l(k, t; \mu_0) - \sin(kx) \Im c_l(k, t; \mu_0)] dk. \end{aligned} \quad (10)$$

Remark IV.1. Although in this section we directly calculated u in (10), indeed, it is possible to directly relate $u(x, \mu, t)$ to $u_1(x, \mu, t)$ which is the solution of (1) with $\alpha = 1$. Let $f_\alpha(t)$ be a function such that

$$(\mathcal{L}f_\alpha)(s) = e^{-s^\alpha}.$$

For example, we have

$$f_{1/2}(t) = \frac{t^{-3/2}}{2\sqrt{\pi}} e^{-1/(4t)}.$$

If we introduce

$$\varphi(\tau, t) = \frac{t}{\alpha \tau^{1+1/\alpha}} f_\alpha\left(\frac{t}{\tau^{1/\alpha}}\right),$$

we have

$$(\mathcal{L}\varphi)(\tau, s) = s^{\alpha-1} e^{-\tau s^\alpha}.$$

Let us consider the Laplace transform of u with respect to s and u_1 with respect to s^α . Assuming $u(x, \mu, 0) = u_1(x, \mu, 0)$, we obtain

$$\left\{ \begin{array}{l} s^\alpha(\mathcal{L}u)(x, \mu, s) - s^{\alpha-1}u(x, \mu, 0) + v\mu\partial_x(\mathcal{L}u)(x, \mu, s) + \sigma_t(\mathcal{L}u)(x, \mu, s) \\ \quad = \sigma_s \int_{-1}^1 p(\mu, \mu')(\mathcal{L}u)(x, \mu', s) d\mu', \\ s^\alpha(\mathcal{L}u_1)(x, \mu, s^\alpha) - u_1(x, \mu, 0) + v\mu\partial_x(\mathcal{L}u_1)(x, \mu, s^\alpha) + \sigma_t(\mathcal{L}u_1)(x, \mu, s^\alpha) \\ \quad = \sigma_s \int_{-1}^1 p(\mu, \mu')(\mathcal{L}u_1)(x, \mu', s^\alpha) d\mu'. \end{array} \right.$$

The above equations imply

$$(\mathcal{L}u)(x, \mu, s) = s^{\alpha-1}(\mathcal{L}u_1)(x, \mu, s^\alpha) = \int_0^\infty u_1(x, \mu, t) s^{\alpha-1} e^{-\tau s^\alpha} d\tau.$$

Therefore u and u_1 are related as

$$u(x, \mu, t) = \int_0^\infty u_1(x, \mu, \tau) \varphi(\tau, t) d\tau.$$

This means that we can obtain u by integrating u_1 , which is the solution of the first-order equation. The solution u is subordinated to the solution u_1 ¹⁷.

V. NUMERICAL CALCULATION

The energy density $U(x, t)$ is introduced as

$$U(x, t) = \int_{-1}^1 u(x, \mu, t) d\mu.$$

Each N gives an approximated value of $U(x, t)$ as

$$U(x, t) \approx U(x, t; N),$$

where

$$U(x, t; N) = \int_{-1}^1 u(x, \mu, t; N) d\mu.$$

We note that $U(x, t) = U(x, t; \infty)$. Let us calculate $U(x, t; N)$ for the initial condition

$$U(x, 0; N) = \delta(x).$$

From (10) we obtain

$$\begin{aligned}
U(x, t; N) &= \int_{-1}^1 \int_{-1}^1 u(x, \mu, t; N) d\mu d\mu_0 \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{ikx} \sum_{n=0}^N |v_n^{(0)}(k)|^2 E_{\alpha}(-\lambda_n(k)t^{\alpha}) dk \\
&= \frac{2}{\pi} \sum_{n=0}^N \int_0^{\infty} |v_n^{(0)}(k)|^2 \\
&\quad \times \left(\cos(kx) \Re E_{\alpha}(-\lambda_n(k)t^{\alpha}) - \sin(kx) \Im E_{\alpha}(-\lambda_n(k)t^{\alpha}) \right) dk.
\end{aligned}$$

In this section we set

$$v = 1, \quad \sigma_a = 0, \quad L = N = 1,$$

and

$$\sigma_s = 10, \quad g = \frac{\beta_1}{3} = 0.9.$$

The matrix $A(k)$ in (7) is given by

$$A(k) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & ik \\ ik & 2k_c \end{pmatrix}.$$

where we introduced

$$k_c := \frac{\sqrt{3}}{2} \sigma_s (1 - g).$$

Its eigenvalues and eigenvectors are obtained as

$$\lambda(k) = \frac{k_c}{\sqrt{3}} \left(1 \pm \sqrt{1 - \left(\frac{k}{k_c} \right)^2} \right),$$

and

$$\mathbf{v}(k) = \frac{1}{\sqrt{\mathcal{N}}} \begin{pmatrix} \frac{ik}{\sqrt{3}} \\ \lambda(k) \end{pmatrix}, \quad \mathcal{N} = \begin{cases} \frac{2k_c^2}{3} \left(1 \pm \sqrt{1 - \left(\frac{k}{k_c} \right)^2} \right), & |k| \leq k_c, \\ \frac{2}{3} k^2, & |k| > k_c. \end{cases}$$

Thus we have

$$|v^{(0)}(k)|^2 = \begin{cases} \frac{1}{2} \left(1 \mp \sqrt{1 - \left(\frac{k}{k_c} \right)^2} \right), & |k| \leq k_c, \\ \frac{1}{2}, & |k| > k_c. \end{cases}$$

The energy density is written as

$$\begin{aligned}
U(x, t; 1) &= \frac{1}{\pi} \int_0^{k_c} \cos(kx) \\
&\times \left[\left(1 - \sqrt{1 - \left(\frac{k}{k_c} \right)^2} \right) E_\alpha \left(-\frac{k_c + \sqrt{k_c^2 - k^2}}{\sqrt{3}} t^\alpha \right) \right. \\
&+ \left. \left(1 + \sqrt{1 - \left(\frac{k}{k_c} \right)^2} \right) E_\alpha \left(-\frac{k_c - \sqrt{k_c^2 - k^2}}{\sqrt{3}} t^\alpha \right) \right] dk \\
&+ \frac{2}{\pi} \int_{k_c}^\infty \cos(kx) \Re E_\alpha \left(-\frac{k_c - i\sqrt{k^2 - k_c^2}}{\sqrt{3}} t^\alpha \right) dk.
\end{aligned} \tag{11}$$

In the diffusion approximation the energy density is given as follows. If the initial condition is given by

$$U_{\text{DA}}(x, 0) = \delta(x),$$

we have^{27,28}

$$\begin{aligned}
U_{\text{DA}}(x, t) &= \frac{1}{\pi} \int_0^\infty \cos(kx) E_\alpha(-D_0 k^2 t^\alpha) dk \\
&= \frac{1}{\sqrt{D_0}} t^{-\frac{\alpha}{2}} M_{\alpha/2} \left(\frac{|x|}{\sqrt{D_0} t^{\alpha/2}} \right),
\end{aligned} \tag{12}$$

where $M_\alpha(z)$ is the M -Wright function defined by

$$M_\alpha(z) := \sum_{n=0}^\infty \frac{(-1)^n z^n}{n! \Gamma(-\alpha(n+1) + 1)}.$$

Equations (11) and (12) are implemented in Fortran. The numerical implementation of the Mittag-Lifter function relies on the algorithm by Gorenflo, Loutchko, and Luchko⁹. Although we saw in §III that $U(x, t)$ asymptotically becomes $U_{\text{DA}}(x, t)$, they are different in general. In Figs. 1 through 3, we plot $U(x, t; 1)$ and $U_{\text{DA}}(x, t)$ for $\alpha = 0.25, 0.5$, and 0.75 , respectively. For all the cases, we see that $U(x, t; 1)$ stays near the source at $x = 0$ for a relatively long time whereas $U_{\text{DA}}(x, t)$ broadens quickly. When $\alpha = 0.75$ we can see that $U(x, t; 1)$ has two peaks. Such a double-peak structure shows up for $\alpha > 1$ in the case of the fractional diffusion equation²⁸. This behavior can be understood from the relation⁵

$$E_\alpha(z) + E_\alpha(-z) = 2E_{2\alpha}(z^2), \quad z \in \mathbb{C}.$$

For sufficiently large k , which corresponds to small x , we asymptotically have³⁵

$$E_\alpha \left(-\frac{k_c - i\sqrt{k^2 - k_c^2}}{\sqrt{3}} t^\alpha \right) \sim \frac{1}{\alpha} \exp \left[\left(-\frac{k_c - i\sqrt{k^2 - k_c^2}}{\sqrt{3}} t^\alpha \right)^{1/\alpha} \right].$$

Hence in (11) we have

$$\begin{aligned}
\Re E_\alpha \left(-\frac{k_c - i\sqrt{k^2 - k_c^2}}{\sqrt{3}} t^\alpha \right) &\sim \frac{1}{2\alpha} \exp \left[\left(i\frac{k}{\sqrt{3}} t^\alpha \right)^{1/\alpha} \right] + \frac{1}{2\alpha} \exp \left[\left(-i\frac{k}{\sqrt{3}} t^\alpha \right)^{1/\alpha} \right] \\
&\sim \frac{1}{2} E_\alpha \left(i\frac{k}{\sqrt{3}} t^\alpha \right) + \frac{1}{2} E_\alpha \left(-i\frac{k}{\sqrt{3}} t^\alpha \right) \\
&= E_{2\alpha} \left(-\frac{1}{3} k^2 t^{2\alpha} \right).
\end{aligned}$$

The above calculation implies that the double-peak behavior for the fractional diffusion equation with $\alpha > 1$ can be seen for the fractional radiative transport equation with $\alpha > 1/2$.

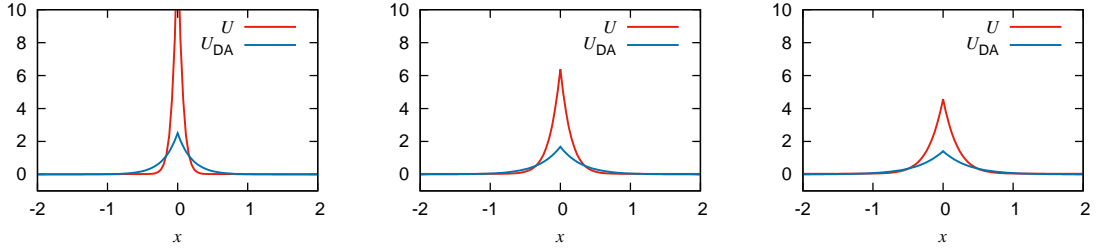


FIG. 1. Comparison of $U(x,t)$ and $U_{\text{DA}}(x,t)$ as a function of x , from the left, for $t = 0.0001$, 0.0025 , and 0.01 , respectively when $\alpha = 0.25$.

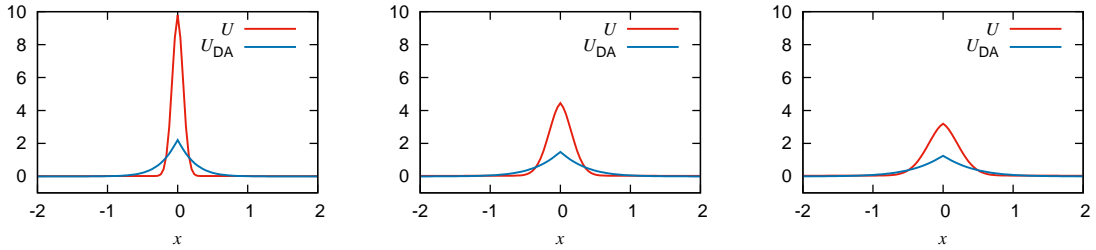


FIG. 2. Comparison of $U(x,t)$ and $U_{\text{DA}}(x,t)$ as a function of x , from the left, for $t = 0.01$, 0.05 , and 0.1 , respectively when $\alpha = 0.5$.

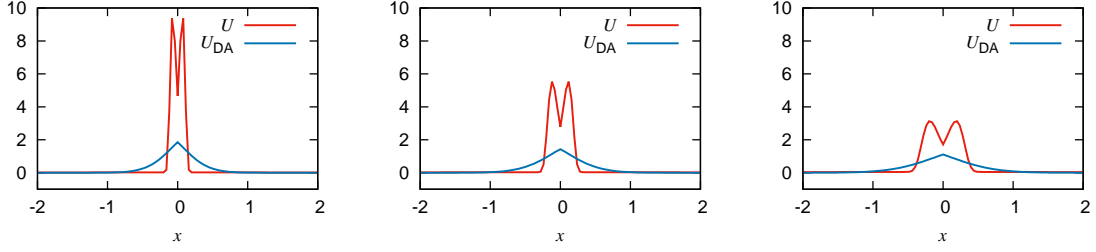


FIG. 3. Comparison of $U(x,t)$ and $U_{\text{DA}}(x,t)$ as a function of x , from the left, for $t = 0.05, 0.1$, and 0.2 , respectively when $\alpha = 0.75$.

VI. CONCLUDING REMARKS

One of the purposes of the present paper is to see the connection between the time-fractional radiative transport equation and the time-fractional diffusion equation. Roughly speaking, the time-fractional radiative transport equation of ∂_t^α behaves as the time-fractional diffusion equation of ∂_t^α for large x and behaves as the time-fractional diffusion equation of $\partial_t^{2\alpha}$ near $x = 0$ as is investigated in §III and §V.

When $u(x, \mu, t)$ in (1) is expressed in the form of the collision expansion, the ballistic term is singular. If $u(x, \mu, t)$ itself is numerically computed, it is desirable to subtract the ballistic term. In a straightforward manner, we can extend the calculation in §IV. This calculation is summarized in Appendix.

Appendix A: Subtraction of the ballistic term

Let us split $u(x, \mu, t)$ in (1) into the ballistic and scattered parts as

$$u(x, \mu, t) = u_b(x, \mu, t) + u_s(x, \mu, t),$$

where $u_b(x, \mu, t)$ and $u_s(x, \mu, t)$ respectively satisfy

$$\begin{cases} \partial_t^\alpha u_b(x, \mu, t) + \mu \partial_x u_b(x, \mu, t) + \sigma_t u_b(x, \mu, t) = 0, \\ u_b(x, \mu, 0) = \delta(x) \delta(\mu - \mu_0), \end{cases}$$

[illegible]
$$S(x, \mu, t; \mu_0) = \sigma_s \int_{-1}^1 p(\mu, \mu') u_b(x, \mu', t) d\mu'.$$
$$(\mathcal{LF}u_b)(k, \mu, s) = \frac{s^{\alpha-1}}{s^\alpha + ik\mu + \sigma_t} \delta(\mu - \mu_0),$$
$$u_b(x, \mu, t) = \frac{1}{2\pi} \delta(\mu - \mu_0) \int_{-\infty}^{\infty} e^{ikx} E_{\alpha} [-(ik\mu_0 + \sigma_t)t^{\alpha}] dk,$$
$$(\mathcal{LFS})(k, \mu, s; \mu_0) = \sigma_s p(\mu, \mu_0) \frac{s^{\alpha-1}}{s^\alpha + ik\mu_0 + \sigma_t}.$$
$$(\mathcal{F}u_s)(k, \mu, t) = \sum_{l=0}^{\infty} \sqrt{2l+1} c_l(k, t; \mu_0) P_l(\mu). \quad (\text{A1})$$
$$A(k)\mathbf{c}(k, t; \mu_0) + \partial_t^\alpha \mathbf{c}(k, t; \mu_0) = \mathbf{w}(k, t; \mu_0),$$
$$\{\mathbf{w}(k, t; \mu_0)\}_l = \frac{\sqrt{2l+1}}{2} \int_{-1}^1 P_l(\mu) (\mathcal{FS})(k, \mu, t; \mu_0) d\mu.$$
$$(\mathcal{L}\mathbf{c})(k, s; \mu_0) = (A(k) + s^\alpha)^{-1} \left[s^{\alpha-1} \mathbf{c}(k, 0; \mu_0) + (\mathcal{L}\mathbf{w})(k, s; \mu_0) \right].$$
$$(\mathcal{L}\mathbf{w})(k, s; \mu_0) = \frac{s^{\alpha-1}}{s^\alpha + ik\mu_0 + \sigma_t} \mathbf{b}(\mu_0),$$

where

$$\{\mathbf{b}(\mu_0)\}_l = \frac{\sigma_s \beta_l}{2\sqrt{2l+1}} \Theta(L-l) P_l(\mu_0).$$

Using the relation

$$= (A(k) + ik\mu_0 + \sigma_t)^{-1} \left(\frac{(A(k) - s^\alpha)^{-1} (\mathcal{L}\mathbf{w})(k, s; \mu_0)}{s^{\alpha-1}} - \frac{s^{\alpha-1}}{s^\alpha - A(k)} \right) \mathbf{b}(\mu_0),$$

we find

$$\begin{aligned} \mathbf{c}(k, t; \mu_0) &= E_\alpha(-A(k)t^\alpha) \mathbf{c}(k, 0; \mu_0) \\ &+ (A(k) + ik\mu_0 + \sigma_t)^{-1} [E_\alpha(-(ik\mu_0 + \sigma_t)t^\alpha) - E_\alpha(A(k)t^\alpha)] \mathbf{b}(\mu_0). \end{aligned}$$

REFERENCES

- ¹Adams, E. E. and Gelhar, L. W., “Field study of dispersion in a heterogeneous aquifer 2. Spatial moments analysis,” *Water Res. Res.* **28**, 3293–3307 (1992).
- ²Arridge, S. R. and Schotland, J. C., “Optical tomography: forward and inverse problems,” *Inverse Problems* **25**, 123010 (2009).
- ³Caputo, M., “Linear model of dissipation whose Q is almost frequency independent-II,” *Geophys. J. R. Astr. Soc.* **13**, 529–539 (1967).
- ⁴Eidelman, S. D. and Kochubei, A. N., “Cauchy problem for fractional diffusion equations,” *J. Diff. Eq.* **199**, 211–255 (2004).
- ⁵Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G., *Higher Transcendental Functions* Vol. 3 (McGraw-Hill, 1955).
- ⁶Fomin, S. A., Chugunov, V. A., and Hashida, T., “Non-Fickian mass transport in fractured porous media,” *Adv. Water Resour.* **34**, 205–214 (2011).
- ⁷Garcia, R. D. M. and Siewert C. E., “On discrete spectrum calculations in radiative transfer,” *J. Quant. Spec. Rad. Trans.* **42**, 385–394 (1989).
- ⁸Gershenson, M., “Time-dependent equation for the intensity in the diffusion limit using a higher-order angular expansion,” *Phys. Rev. E* **59**, 7178–7184 (1999).
- ⁹Gorenflo, R., Loutchko, J., and Luchko, Y., “Computation of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ and its derivative,” *Fract. Calc. Appl. Anal.* **5**, 491–518 (2002).
- ¹⁰Gorenflo, R., Luchko, Y., and Yamamoto, M., “Time-fractional diffusion equation in the fractional Sobolev spaces,” *Fract. Calc. Appl. Anal.* **18**, 799–820 (2015).

- ¹¹Hatano, Y. and Hatano, N., “Dispersive transport of ions in column experiments: An explanation of long-tailed profiles,” *Water Resour. Res.* **34**, 1027–1033 (1998).
- ¹²Henry, B. I., Langlands, T. A. M., and Wearne, S. L., “Anomalous diffusion with linear reaction dynamics: From continuous time random walks to fractional reaction-diffusion equations,” *Phys. Rev. E* **74**, 031116 (2006).
- ¹³Hornung, G., Berkowitz, B., and Barkai, N., “Morphogen gradient formation in a complex environment: An anomalous diffusion model,” *Phys. Rev. E* **72**, 041916 (2005).
- ¹⁴Jin, B. and Rundell, W., “A tutorial on inverse problems for anomalous diffusion processes,” *Inverse Problems* **31**, 035003 (2015).
- ¹⁵Kadem, A., Luchko, Y., and Baleanu, D., “Spectral method for solution of the fractional transport equation,” *Rep. Math. Phys.* **66**, 103–115 (2010).
- ¹⁶Kochubei, A. N., “Distributed order calculus and equations of ultraslow diffusion,” *J. Math. Anal. Appl.* **340**, 252–281 (2008).
- ¹⁷Langlands, T. A. M., Henry, B. I., and Wearne, S. L., “Fractional cable equation models for anomalous electrodiffusion in nerve cells: infinite domain solutions,” *J. Math. Biol.* **59**, 761–808 (2009).
- ¹⁸Larsen, E. W. and Keller, J. B., “Asymptotic solution of neutron transport problems for small mean free paths,” *J. Math. Phys.* **15**, 75–81 (1974).
- ¹⁹Li, Z., Liu, Y., and Yamamoto, M., “Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients,” *Appl. Math. Comp.* **257**, 381–397 (2015).
- ²⁰Li, Z., Luchko, Y., and Yamamoto, M., “Asymptotic estimates of solutions to initial-boundary-value problems for distributed order time-fractional diffusion equations,” *Fract. Cal. Appl. Anal.* **17**, 1114–1136 (2014).
- ²¹Liemert, A. and Kienle, A., “Infinite space Green’s function of the time-dependent radiative transfer equation,” *Biomed. Opt. Exp.* **3**, 543–551 (2012).
- ²²Lin, Y. and Xu, C., “Finite difference/spectral approximations for the time-fractional diffusion equation,” *J. Comp. Phys.* **225**, 1533–1552 (2007).
- ²³Luchko, Y., “Maximum principle for the generalized time-fractional diffusion equation,” *J. Math. Anal. Appl.* **351**, 218–223 (2009).
- ²⁴Luchko, Y., “Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation,” *Comp. Math. Appl.* **59**, 1766–

- 1772 (2010).
- ²⁵Luchko, Y., “Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation,” *J. Math. Anal. Appl.* **374**, 538–548 (2011).
 - ²⁶Luchko, Y., “Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation,” *Fract. Cal. Appl. Anal.* **15**, 141–160 (2012).
 - ²⁷Mainardi, F., “The fundamental solutions for the fractional diffusion-wave equation,” *Appl. Math. Lett.* **9**, 23–28 (1996).
 - ²⁸Mainardi, F., Luchko, Y., Pagnini, G., “The fundamental solution of the space-time fractional diffusion equation,” *Fract. Cal. Appl. Anal.* **4**, 153–192 (2001).
 - ²⁹Mellet, A., “Fractional diffusion limit for collisional kinetic equations: A moments method,” *Indiana Univ. Math. J.* **59**, 1333–1360 (2010).
 - ³⁰Mellet, A., Mischler, S., and Mouhot, C., “Fractional diffusion limit for collisional kinetic equations,” *Arch. Rational Mech. Anal.* **199**, 493–525 (2011).
 - ³¹Metzler, R. and Klafter, J., “The random walk’s guide to anomalous diffusion: a fractional dynamics approach,” *Phys. Rep.* **339**, 1–77 (2000).
 - ³²Metzler, R. and Klafter, J., “The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics,” *J. Phys. A: Math. Gen.* **37**, R161–R208 (2004).
 - ³³Metzler, R., Jeon, J.-H., Cherstvy, A. G., and Barkaid, E., “Anomalous diffusion models and their properties: non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking,” *Phys. Chem. Chem. Phys.* **16**, 24128–24164 (2014).
 - ³⁴Panasyuk, G., Schotland, J. C., and Markel, V. A., “Radiative transport equation in rotated reference frames,” *J. Phys. A: Math. Gen.* **39**, 115–137 (2006).
 - ³⁵Podlubny, I., *Fractional Differential Equations* (Academic Press, 1999).
 - ³⁶Ryzhik, L., Papanicolaou, G., and Keller, J. B., “Transport equations for elastic and other waves in random media,” *Wave Motion* **24**, 327–370 (1996).
 - ³⁷Sakamoto, K. and Yamamoto, M., “Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems,” *J. Math. Anal. Appl.* **382**, 426–447 (2011).
 - ³⁸Samko, S. G., Kilbas, A. A., and Marichev, O. I., *Fractional integrals and derivatives: theory and applications* (Gordon and Breach Science, 1993).
 - ³⁹Sokolov, I., Klafter, J., and Blumen, A., “Fractional Kinetics,” *Physics Today* **55**, 48–54

(2002).

⁴⁰Williams, M. M. R., “Stochastic problems in the transport of radioactive nuclides in fractured rock,” *Nucl. Sci. Eng.* **112**, 215–230 (1992).

⁴¹Williams, M. M. R., “Radionuclide transport in fractured rock a new model: application and discussion,” *Ann. Nucl. Energy* **20**, 279–297 (1993).